# Assignment 3: GMM Exercise

Problem 1

A canonical application of the EM algorithm is its use in fitting a mixture model where we assume we observe an i.i.d. sample of $x_1, \dots x_n$ from

$$Y \sim \text{Multinomial}(1, p_1, \dots, p_k)$$

$$X|Y = l \sim \mathcal{N}(\mu_l, \sigma_l^2)$$

with the simplest example of $\mathcal{N}(\mu_l, \sigma_l^2)$ being the univariate normal model.

1. Show that the joint distribution of $(X, Y)$ is an exponential family;
2. What is the marginal density of $X$?
3. Write out the log-likelihood $LL(\Theta|X)$ based on observing an i.i.d. sample $x_1, \dots x_n$ from this model. What are its parameters?

Problem 2 (EM for a 1D Laplacian Mixture Model)

In this problem you will derive the EM algorithm for a one-dimensional Laplacian mixture model. You are given $n$ observations $x_1, \dots, x_n \in \mathbb{R}$ and we want to fit a mixture of $m$ Laplacians, which has the following density

$$f(x) = \sum_{j=1}^{m} \alpha_j f_L(x; \mu_j, \beta_j)$$

where $f_L(x; \mu_j, \beta_j) = \frac{1}{2\beta_j} e^{-\frac{1}{\beta_j}|x-\mu_j|}$, and the mixture weights $\alpha_j$ are a convex combination, i.e. $\alpha_j \geq 0$ and $\sum_{j=1}^{m} \alpha_j = 1$. For simplicity, assume that the scale parameters $\beta_j > 0$ are known beforehand and thus fixed.

1. Introduce latent variables so that we can apply the EM procedure.
2. Analogously to the previous question, write down the steps of the EM procedure for this model. If some updates cannot be written analytically, give an approach on how to compute them.